

Dihedral Groups: The group theory of isometries of n -sided polygons

Additional terminology: singular complex number and full-rank complex number

These paragraphs were for the next document that I am in the process of writing but is required with this one so I have decided to move it here:

Since $\hat{k} \times (\cdot)$ and $\exp\left(\frac{1}{2}\pi \hat{k} \times\right)(\cdot)$ are not equivalent inasmuch as the former is singular (rank 2 nullity 1) whilst the latter is full rank (rank 3 nullity 0), this allows for some additional terminology to distinguish between the two:

- $\hat{k} \times (\cdot)$ can be termed the **singular complex number** (for the x-y plane), since it has rank 2 and nullity 1, whilst
- $\exp\left(\frac{1}{2}\pi \hat{k} \times\right)(\cdot)$ can be termed the **full-rank complex number** (for the x-y plane), since it has rank 3 and nullity 0.

The full rank complex number can be written as $\hat{k} \hat{k}' + \hat{k} \times$ which is the singular complex number ($\hat{k} \times$) with the addition of the rotational eigenvalue ($\hat{k} \hat{k}'$). The addition of the eigenvalue removes the nullity and hence gives it full rank, i.e., increases its rank from 2 to 3 in the R^3 basis system. As a reminder

$$\hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (rref has 2 pivots)} \quad \exp\left(\frac{1}{2}\pi \hat{k} \times\right) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ (rref has 3 pivots)} \quad \hat{k} \hat{k}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ (rref has 1 pivot)}$$

(rref = Row Reduced Echelon Form) where $\hat{k} = (0 \ 0 \ 1)'$ is the k -hat unit basis vector [and when appended with $\times$ is the **singular complex number**]. The terminology of **singular complex number** and **full rank complex number** makes it clearer when one or the other is more appropriate for a particular circumstance. For example, the complex i in phasor calculations on a calculator would probably be best programmed in terms of the **full rank-complex number**. For differential equations, on the other hand, if the **full rank complex number** is used the rotational eigenvalue gets double counted which would lead to wrong results and hence the **singular-complex number** is required [this is going to lead to my next set of discoveries of many new equations]. From what I know about differential equations [most notably that just mentioned] and how the concept of complex numbers has worked in these concepts, I can state confidently that the **singular complex number** is that which corresponds best to what was hence previously known as i [that is, $i = \hat{k} \times$]. Further examples of where the **full rank complex number** is required, rather than the **singular complex number**, is dihedral groups which is the topic of this document. These require the **full rank complex number** similarly as **Rubiks cubes** and **Rubiks derivatives**, and it is conjectured that this is because, in molecular symmetries and spectroscopy, these correspond to solutions of differential equations [of bonding, valence and non-valence electrons (which are basis-exponentials)] which convert the **singular complex number** into the **full rank complex number**. Note that what others commonly call **twisty puzzles** is what I call **Rubiks derivatives** [partially because of my finance PhD (unpassed) in financial derivatives and so I know that it is a good name for the same reason as **financial derivatives** (**deferred trades of [derived from] spot market transactions [topic of my PhD]**) and **carboxylic acid derivatives** (esters, anhydrides, acid chlorides, and amides derived from organic chemistry's main acid COOH)].

Whilst [in my opinion] the theory of **single component basis cross products** replaces complex numbers [since there is no such thing], I still fall back on complex numbers when I need some hard and fast algebra, although I now commonly replace i with k -cross as a personal preference for mathematical style, especially in exponentials... I have also started appending cross $\times$ to j in electrical engineering applications also. In doing so, the primary consideration is whether the singular or full-rank basis curl (k -cross for example) is more appropriate.

An item of relevance is that **rotation matrices** (**basis exponentials**) are **full rank** complex-numbers/quaternions whilst **cross-products** (linear combinations of i -cross, j -cross, and k -cross) are sets of **singular** complex-numbers/quaternions. This should make it clearer when one or the other is more appropriate. I may also have implications for the curl of curl, nabla-double-cross.

Also note that some further linear algebra properties of k -cross are that k -cross is **symmetric** and **periodic** since

$$(\hat{k} \times)(\hat{k} \times)^T = I \quad \text{and} \quad (\hat{k} \times)^4 = I$$

This is a consequence of k -cross being the rectangular complex number for the x-y plane.

k -hat eigenvalue basis exponential

Noting that

$$(\hat{k} \hat{k}')^2 = (\hat{k} \hat{k}') \hat{k} \hat{k}' = \hat{k} (\hat{k} \hat{k}') \hat{k}' = \hat{k} (1) \hat{k}' = \hat{k} \hat{k}'$$

$$(\hat{k} \hat{k}')^3 = (\hat{k} \hat{k}')^2 \hat{k} \hat{k}' = (\hat{k} \hat{k}') \hat{k} \hat{k}' = \hat{k} \hat{k}'$$

$$(\hat{k} \hat{k}')^n = (\hat{k} \hat{k}')^{n-1} = \dots = \hat{k} \hat{k}'$$

in general, then the exponential of the rotational eigenvalue basis component is

$$\begin{aligned} e^{\theta \hat{k} \hat{k}'} &= 1 + (\theta \hat{k} \hat{k}') + \frac{1}{2!} (\theta \hat{k} \hat{k}')^2 + \frac{1}{3!} (\theta \hat{k} \hat{k}')^3 + \dots + \frac{1}{n!} (\theta \hat{k} \hat{k}')^n + \dots \\ &= 1 + \theta (\hat{k} \hat{k}') + \frac{1}{2!} \theta^2 (\hat{k} \hat{k}')^2 + \frac{1}{3!} \theta^3 (\hat{k} \hat{k}')^3 + \dots + \frac{1}{n!} \theta^n (\hat{k} \hat{k}')^n + \dots \\ &= 1 + \theta (\hat{k} \hat{k}') + \frac{1}{2!} \theta^2 (\hat{k} \hat{k}') + \frac{1}{3!} \theta^3 (\hat{k} \hat{k}') + \dots + \frac{1}{n!} \theta^n (\hat{k} \hat{k}') + \dots \quad \text{since } (\hat{k} \hat{k}')^n = \hat{k} \hat{k}' \\ &= 1 + \left(\theta + \frac{1}{2!} \theta^2 + \frac{1}{3!} \theta^3 + \dots + \frac{1}{n!} \theta^n + \dots \right) (\hat{k} \hat{k}') \\ &= 1 + \left(-1 + 1 + \theta + \frac{1}{2!} \theta^2 + \frac{1}{3!} \theta^3 + \dots + \frac{1}{n!} \theta^n + \dots \right) (\hat{k} \hat{k}') \\ &= 1 + (e^\theta - 1) \hat{k} \hat{k}' \end{aligned}$$

For the derived formula, $e^{\theta \hat{k} \hat{k}'} = 1 + (e^\theta - 1) \hat{k} \hat{k}'$, in each basis direction, i -hat, j -hat, and k -hat

$$e^{\theta \hat{k} \hat{k}'} \hat{i} = \hat{i} + (e^\theta - 1) \hat{k} \hat{k}' \hat{i} = \hat{i}$$

$$e^{\theta \hat{k} \hat{k}'} \hat{j} = \hat{j} + (e^\theta - 1) \hat{k} \hat{k}' \hat{j} = \hat{j}$$

$$e^{\theta \hat{k} \hat{k}'} \hat{k} = \hat{k} + (e^\theta - 1) \hat{k} \hat{k}' \hat{k} = e^\theta \hat{k}$$

Expanding diadically results in the k -hat eigenvalue basis exponential as an operator (for a **position** or **colour** vector)

$$e^{\theta \hat{k} \hat{k}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^\theta \end{pmatrix}$$

Since I have not commuted theta in the derivation, theta can be scalar or can take the form of i -cross or j -cross (k -cross returns the identity). This is where a complex [basis] exponential for i -hat (y - z plane) or j -hat (x - z plane) is placed on the k -hat component of the diagonal which then expands into a scalar sine (on the i -hat or j -hat row) and scalar cosine (on the k -hat row) in that column (the k -hat column). Of course, the exponential can also be utilised as attenuation for the k -hat axis by keeping theta scalar. This formula will also contribute to making it clearer how to avoid double counting an eigenvalue when solving differential equations.

Introduction

Dihedral groups are a form of applied (rather than abstract) group theory and are mathematical constructs utilized in molecular shape and spectroscopy, especially in x-ray crystallography. The theory of point groups determines which absorbances will show up in the various region of the electromagnetic spectrum that is excited. All of these various forms of spectroscopy correspond to excitation of **basis-exponentials** [a new term I introduced (for the k -hat, j -hat, and i -hat exponentials) given that there is no such thing as a complex field... introduced probably as a complement rather than substitute or replacement, for the term **complex exponential**... k -hat, j -hat, and i -hat exponentials are the complex exponentials]. In the theory of point groups [chemistry and physics] and dihedral groups [mathematics], these notions correspond to the basis-crosses whereby the singular complex numbers have their rotational eigenvalues added converting them to the full rank complex numbers [which is, ultimately, a result of solving a differential equation]. This corresponds to exponentiating the basis-exponentials at discrete values over a given period/wavelength. I hence postulate that all of the dihedral groups, such as those in molecular spectroscopy and x-ray crystallography, are **systems of basis-exponentials** exactly similarly as **Rubiks cubes** and **Rubiks derivatives**. As for k -cross (for complex numbers) and **colour vectors** (for Rubiks cubes), I think I demonstrate this successfully [it is partially known already in this case]. This is the frontiers of chemistry, physics, and pure applied mathematics. It was also a subtopic of the document that I titled ' **k -cross and the k -hat exponential**' and the document that added to that which introduced the concept of **colour vectors** to keep track of both **position** and **orientation** of bits of a Rubiks cube.

It is also a fact that almost all of Digital Signal Processing (DSP) is mathematics of basis-exponentials, mostly in the form of difference equations. This is more credence to the idea that everything is perfect circles which now in the modern terminology is that everything consists of basis exponentials. I believe that this is so because I believe that there is a Maxwell basis [since waves require a basis to propagate] and this Maxwell basis consists of basis exponentials [as per solutions to Maxwell equations]. Calculus dynamics then scales and combines these [with dees and inverse dees (poles and zeros)] into the macroscopic effects that are experienced by our senses [and instruments], that is, felt, seen, and heard as heat, light, and sound.

I will start with the dihedral group corresponding to square planar **D_4** .

Dihedral group of a square (square planar) D_4

I'll first address why I chose to do a square first. k -cross corresponds to the quartic roots of unity which is the order of the rotations of a square. Since the dihedral group of a square is a group of order 8, the rotation elements corresponds to an order 4 subgroup of the 8 elements in this group. These are the simplest elements that are self-apparent from complex numbers and hence why I chose to do a square first. This is the cyclic group of k -cross which is the 'axis of proper rotation' for the square and the order 4 subgroup of the full order 8 group of the D_4 . The axis for the rotational subgroup, can either be k -hat, j -hat, or i -hat, that is, j -cross or i -cross rather than k -cross. Since a dihedral polygon is usually depicted on the page corresponding the x - y plane (generated by i -hat and j -hat), the 'axis of proper rotation' is chosen as k -hat for the exact same reason as for the theory of k -cross as the true complex number.

I haven't initially worried about the rotational eigenvector; because it is a plane, the k -hat axis is the rotational axis and can hence be ignored as latent for the plane in R^3 similarly as complex numbers (if there were such a thing). Therefore, the rotational eigenvector can be ignored. It does not matter if it is singular or full rank. Expressing the quartic roots of unity in terms of the **singular complex number** k -cross is simpler. These are

$$(1) \hat{k} \times \quad (2) (\hat{k} \times)^2 \quad (3) (\hat{k} \times)^3 \quad (4) (\hat{k} \times)^4$$

the last being the identity for the x - y plane, although it should be pointed out, not identity for the R^3 volume since the diadically expanded operator lacks the kk' component of the identity matrix [so it has only two of the three ones required for the diagonal]. The other 4 elements of the cyclic group of order 8, D_4 , correspond to reflection and reflection plus rotation and act as a mirror plane. The reflection can be around either of the remaining principal axes, either j -hat or i -hat, or any other axis that is chosen that symmetrically bisects the square. I have decided to rotate around j -hat (with j -cross) as the initial axis for pure reflection of a mirror plane. The other mirror planes are then i -hat, or any set of orthogonal axes for that plane such as $(\hat{i} + \hat{j})/\sqrt{2}$ or $(\hat{i} - \hat{j})/\sqrt{2}$. The easier way though, is to choose j -hat for the first reflection, and then apply k -cross in succession for the other reflections plus rotations. This demonstrates again that there are several ways to define the operators [for reflection plus rotation] and this is for the same reason as minimal polynomials for characteristic equations and why row reducing a matrix in pivotal condensation (with k -cross for example) can be performed in many different ways. Taking j -hat as the reflection, the remaining 4 elements of D_4 are first considered by rotating around j -hat with singular j -hat double-cross,

$$(5) \hat{j} \times \hat{j} \times$$

and the remaining three elements being

$$(6) \hat{k} \times \hat{j} \times \hat{j} \times \quad (7) (\hat{k} \times)^2 \hat{j} \times \hat{j} \times \quad (8) (\hat{k} \times)^3 \hat{j} \times \hat{j} \times \quad \text{noting obviously that } (\hat{k} \times)^4 \hat{j} \times \hat{j} \times = \hat{j} \times \hat{j} \times$$

I knew that the theory of k -cross and the k -hat exponential could easily model dihedral groups, and so this was my first guess for the dihedral group of order 4, the symmetry elements of a square polygon which has $2n = 8$ elements (2 faces for the dihedral group times the 4 sides of the polygon which is also the cyclic [periodic] order of k -cross). Therefore my first guess was the set of 8 elements

$$\begin{array}{llll} (1) \hat{k} \times & (2) (\hat{k} \times)^2 & (3) (\hat{k} \times)^3 & (4) (\hat{k} \times)^4 \\ (5) \hat{k} \times (\hat{j} \times)^2 & (6) (\hat{k} \times)^2 (\hat{j} \times)^2 & (7) (\hat{k} \times)^3 (\hat{j} \times)^2 & (8) (\hat{k} \times)^4 (\hat{j} \times)^2 \end{array}$$

where $(\hat{k} \times)^4$ is identity (for R^2 but not R^3) and $(\hat{k} \times)^4 (\hat{j} \times)^2$ is pure reflection (reflection without rotation) of j -double-cross.

But the last four (4)-(8) are not correct because, unlike the k -hat axis, the j -hat axis is not latent (since it is k -hat's orthogonal complement). Hence the lack of full rank in the singular operator j -cross introduces nilpotence. This means that the required cyclic nature of the operator [required for the group] breaks down. This is overcome simply by introducing the rotational eigenvalue and eigenvector for j -hat to remove the nullity and so make j -cross full rank for the reflection. Here was a decision point for me because there are two ways of doing this [actually two and a half ways], that is, two ways of introducing the rotational eigenvalue for the j -hat reflections [since j -hat is not latent for the x - y plane]. This is done either by (1) exponentiating and scaling by $\pi/2$, that is, by utilizing the j -hat basis exponential or (2) adding the matrix basis component for the rotational eigenvalue directly.

Since I was still working this out, it was not clear which is best: exponentiate for a $\pi/2$ rotation or just add the rotational eigenvalue to j -cross. Whether to just add the rotational eigenvalue to j -cross (and then square) or j -double-cross is a small paradox of a sort [the "half" in "two and a half ways of doing this"]. Since the k -hat axis is latent, it doesn't matter for this rotation axis so either k -cross or the k -hat exponential works just fine. But not for j -cross, or as required here for j -double-cross. Taking the approach of adding the matrix basis component jj' for the unity rotational eigenvalue, do I (1) add the rotational eigenvalue directly to j -cross and then square for the full rank double-cross or (2) square the double-cross and then add the rotation eigenvalue? Turns out here it doesn't matter, both work... [which is interesting]... This is because the eigenvalue for j -hat rotation as a matrix basis component, multiplied with j -cross is nilpotent. This may also be a reason why Gauss density is so prominent in physical systems, in addition to the two other reasons, those of the Central Limit Theorem and the hypothesized Maxwell basis [so that any linear combination or scaling up to macroscopic properties from the Maxwell basis is also Gauss density]. This reason is that exponentiation of the double-cross, when

scaled, is Gauss density, which again, is interesting. Anyway, back to the point at hand which is the reflection operator for the singular j -double-cross with the rotational eigenvalue to make it full rank: is it equal to

$$(\hat{j}\hat{j}' + \hat{j} \times)^2 \quad \text{or} \quad \hat{j}\hat{j}' + (\hat{j} \times)^2$$

Turns out that it doesn't matter, because nilpotence in the non-commutative cross terms in the former mean that they are both the same

$$(\hat{j}\hat{j}' + \hat{j} \times)^2 = (\hat{j}\hat{j}' + \hat{j} \times)(\hat{j}\hat{j}' + \hat{j} \times) = (\hat{j}\hat{j}')^2 + (\hat{j}\hat{j}')(\hat{j} \times) + (\hat{j} \times)(\hat{j}\hat{j}') + (\hat{j} \times)^2 = \hat{j}\hat{j}' + (\hat{j} \times)^2$$

$$\text{since } (\hat{j}\hat{j}')^2 = \hat{j}\hat{j}'\hat{j}\hat{j}' = \hat{j}(\hat{j}'\hat{j})\hat{j}' = \hat{j}(1)\hat{j}' = \hat{j}\hat{j}' \quad \text{and} \quad (\hat{j} \times)(\hat{j}\hat{j}') = \hat{j} \times \hat{j}\hat{j}' = (\hat{j} \times \hat{j})\hat{j}' = (0\hat{j})\hat{j}' = 0\hat{j}\hat{j}'$$

whilst for $(\hat{j}\hat{j}')(\hat{j} \times)$, in the i -hat direction

$$(\hat{j}\hat{j}')(\hat{j} \times)\hat{i} = (\hat{j}\hat{j}')(\hat{j} \times \hat{i}) = -(\hat{j}\hat{j}')\hat{k} = -\hat{j}\hat{j}'\hat{k} = -\hat{j}(\hat{j}'\hat{k}) = -\hat{j}(0) = 0$$

the j -hat direction

$$(\hat{j}\hat{j}')(\hat{j} \times)\hat{j} = (\hat{j}\hat{j}')(\hat{j} \times \hat{j}) = (\hat{j}\hat{j}')(0\hat{j}) = 0\hat{j}\hat{j}'\hat{j} = 0\hat{j}(1) = 0\hat{j} = 0$$

and k -hat direction

$$(\hat{j}\hat{j}')(\hat{j} \times)\hat{k} = (\hat{j}\hat{j}')(\hat{j} \times \hat{k}) = (\hat{j}\hat{j}')\hat{i} = \hat{j}\hat{j}'\hat{i} = \hat{j}(\hat{j}'\hat{i}) = \hat{j}(0) = 0$$

and so is null for each basis direction in an operand. The results for $(\hat{j}\hat{j}')^2$, $(\hat{j} \times)(\hat{j}\hat{j}')$, and $(\hat{j}\hat{j}')(\hat{j} \times)$, and hence the equivalence of $(\hat{j}\hat{j}' + \hat{j} \times)^2$ and $\hat{j}\hat{j}' + (\hat{j} \times)^2$, are verified by diadic expansions and working in matrix bases which I will do here shortly... I'll do it now actually

$$\hat{j}\hat{j}' + \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{so} \quad (\hat{j}\hat{j}' + \hat{j} \times)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\hat{j}\hat{j}' + (\hat{j} \times)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{hence } (\hat{j}\hat{j}' + \hat{j} \times)^2 = \hat{j}\hat{j}' + (\hat{j} \times)^2$$

i.e., $(\hat{j}\hat{j}' + \hat{j} \times)^2 = \hat{j}\hat{j}' + \hat{j} \times \hat{j} \times$ so **(1)** the square of the full rank j -cross equals **(2)** the full rank j -hat double-cross. All of this theory works equivalently with i -hat-double-cross rather than j -hat-double-cross to get the reflection for the second set of 4 rotation elements since they are the interchangeable principal axes for the plane of the dihedral group. Many other combinations are also possible for the axis of reflection which utilize both i -hat-double-cross (i -div) and j -hat-double-cross (j -div), so long as they are an orthogonal spanning set which form the orthogonal complement of the k -hat axis of proper rotation. The k -cross (k -curl) is obviously necessary though, for the rotation axis, that is, for the axis of proper rotation in molecular group theory.

Interestingly, because of nilpotence, the arguments above proving that $(\hat{j}\hat{j}' + \hat{j} \times)^2 = \hat{j}\hat{j}' + (\hat{j} \times)^2$ hold in general for arbitrary positive powers. **(1)** The powers of the full rank k -cross, that with the rotational eigenvalue, mathematically simplify to **(2)** powers of the singular k -cross plus the rotational eigenvalue. This is an interesting formula. It is probably a significant reason why a complex number can be manipulated as though it is purely algebraic [because the rotational eigenvalue can be added (or removed) at any stage in raising to powers]. This formula, the generalization for that of full rank j -double-cross, is

$$(\hat{j}\hat{j}' + \hat{j} \times)^n = \hat{j}\hat{j}' + (\hat{j} \times)^n$$

and similarly

$$(\hat{k}\hat{k}' + \hat{k} \times)^n = \hat{k}\hat{k}' + (\hat{k} \times)^n \quad \text{and} \quad (\hat{i}\hat{i}' + \hat{i} \times)^n = \hat{i}\hat{i}' + (\hat{i} \times)^n \quad \text{for any positive } n, \text{ rational or otherwise.}$$

Since it is an equivalent to the basis exponential at $\pi/2$ times n , it is hence another formula for the basis exponential and equivalently the rotation matrix (with argument $\pi n/2$). The single caveat is that $n > 0$, that is, for positive angles (counter-clockwise-rotations). The number n is the number of right angles by which the operator rotates the operand ($n=4$ corresponds to 360°). Examples for this equation include the k -hat-triple-cross $\hat{k} \hat{k}' + \hat{k} \times \hat{k} \times \hat{k} \times$ and the first of the full rank octal roots of unity $\hat{k} \hat{k}' + \sqrt[4]{\hat{k} \times}$, that is, $\hat{k} \hat{k}' + (\hat{k} \times)^{4/8}$.

$(\hat{k} \hat{k}' + \hat{k} \times)^n = \hat{k} \hat{k}' + (\hat{k} \times)^n$ is an interesting formula for the discussion of the distinction between singular and full rank complex number. This means that de-Moivre's theorem for non-negative powers holds [subject to adding back the rotational eigenvalue] for either the singular or the full rank complex number since the eigenvalue can be added to the singular complex number at any time (after any raising to powers) in contrast to the full-rank complex number that has it structurally added at the outset with **cosk** (the k -hat cosine). Conversely, the basis cosine **cosk** necessarily provides the rotational eigenvalue to the basis exponential at the outset whilst one can do the same mathematics with the singular complex number and then add back the eigenvalue once the powers are complete, or at any other stage, making the two equivalent [with the allowable addition of kk]. It also makes it clear that one cannot just multiply out the singular basis-crosses [the rotations combined with the reflections], as for example for quaternions [which would require that i, j , and k be full rank]. This probably explains why quaternions haven't made it into mainstream mathematics in the same way as the algebraic complex number i has, that is, why quaternions are not ubiquitous as algebra and calculus in i . More specifically this is because when based on only one plane (on axis latent) it does not matter whether the rotation is singular or full rank; it does in the context of quaternions whereby each of three plane in R^3 are recognised... this theory makes it clear how to make it all work in R^3 [that is, in quaternion theory for example]. This may also have implications for curl-curl, that is the curl of curl (nabla-double-cross) since nabla is an operator which is a function of the singular complex numbers. The products of the singular complex numbers may have introduced unrecognised or unwanted nilpotence in a particular application. The addition of a constant to nabla before appending cross for vector calculus curl (as discussed in **k-cross and the k-hat exponential**) makes each component full rank (if that is what is required for example).

I could have just started with the full rank operators, but I think starting with the singular operators [which was the starting point for my reasoning] provides additional relevant detail so that the theory can be competently applied [as a reader or researcher may find themselves at a decision point as to when a singular or full rank complex number is more appropriate as I did when working this out]. The fact that the rotational eigenvector is required suggests to me that these may well eventually be better expressed as differential equations. It is also how I found the interesting formula for basis exponentials [and hence equivalently for rotation matrices].

So although I could leave out the rotational eigenvalue for k -cross, I now include it so that it is consistent with the full rank j -cross and j -double-cross (the square of j -cross), and also if one wants to do additional things with the polygon, such as only partially reflecting it or whatever, it is all possible once latent k -hat in R^3 is recognized. It is definitely better with the inclusion of the rotational eigenvalue for k -cross, since it is required for j -cross as j -double-cross [although now it can be added before or after the squaring]. I left it out at the start just to make the point. This provides some additional heuristics/experience as to when the full rank complex number is required as apposed to the singular complex number [it wasn't clear at the very beginning of this that it is needed for the reflections although it became clear pretty early on and it does seem obvious with hindsight].

The $2n = 2 \times 4 = 8$ symmetry operators D_4 comprising the **four rotations** and **four reflections with rotation**, that is, **proper rotations** $\sigma_4^1 \sigma_4^2 \sigma_4^3 \sigma_4^4$ and **rotations after reflection** $r_4^1 r_4^2 r_4^3 r_4^4$, are

$$\begin{aligned} (1) \quad & \hat{k} \hat{k}' + \hat{k} \times & (2) \quad & (\hat{k} \hat{k}' + \hat{k} \times)^2 & (3) \quad & (\hat{k} \hat{k}' + \hat{k} \times)^3 & (4) \quad & (\hat{k} \hat{k}' + \hat{k} \times)^4 \\ (5) \quad & (\hat{k} \hat{k}' + \hat{k} \times)(\hat{j} \hat{j}' + \hat{j} \times)^2 & (6) \quad & (\hat{k} \hat{k}' + \hat{k} \times)^2(\hat{j} \hat{j}' + \hat{j} \times)^2 & (7) \quad & (\hat{k} \hat{k}' + \hat{k} \times)^3(\hat{j} \hat{j}' + \hat{j} \times)^2 & (8) \quad & (\hat{k} \hat{k}' + \hat{k} \times)^4(\hat{j} \hat{j}' + \hat{j} \times)^2 \end{aligned}$$

In terms of the formula for powers of the singular complex numbers

$$\begin{aligned} (1) \quad & \hat{k} \hat{k}' + \hat{k} \times & (2) \quad & \hat{k} \hat{k}' + (\hat{k} \times)^2 & (3) \quad & \hat{k} \hat{k}' + (\hat{k} \times)^3 & (4) \quad & \hat{k} \hat{k}' + (\hat{k} \times)^4 \\ (5) \quad & (\hat{k} \hat{k}' + \hat{k} \times)(\hat{j} \hat{j}' + (\hat{j} \times)^2) & (6) \quad & (\hat{k} \hat{k}' + (\hat{k} \times)^2)(\hat{j} \hat{j}' + (\hat{j} \times)^2) & (7) \quad & (\hat{k} \hat{k}' + (\hat{k} \times)^3)(\hat{j} \hat{j}' + (\hat{j} \times)^2) & (8) \quad & (\hat{k} \hat{k}' + (\hat{k} \times)^4)(\hat{j} \hat{j}' + (\hat{j} \times)^2) \end{aligned}$$

The operators can also be expressed in terms of (1) the basis-exponentials to which they correspond, which is equivalent as above in that taking the exponent of a singular basis-cross adds the rotational eigenvalues (k -hat- k -hat-prime and j -hat- j -hat-prime) making the operators non-singular and equivalent, or as (2) diadically expanded matrix operators which makes short work of the mathematics with a programmable calculator [probably on standard calculators as well soon]. Makes short work of the character tables as well, which I'll do after this. The eight **cisk** and **cisjj** symmetry operators (proper rotations $\sigma_4^1 \sigma_4^2 \sigma_4^3 \sigma_4^4$ and reflections $r_4^1 r_4^2 r_4^3 r_4^4$) as basis exponentials are

$$(1) \quad \exp\left(\frac{1}{2}\pi \hat{k} \times\right) \quad (2) \quad \exp\left(\frac{2}{2}\pi \hat{k} \times\right) \quad (3) \quad \exp\left(\frac{3}{2}\pi \hat{k} \times\right) \quad (4) \quad \exp\left(\frac{4}{2}\pi \hat{k} \times\right)$$

$$(5) \exp\left(\frac{1}{2}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times) \quad (6) \exp\left(\frac{2}{2}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times) \quad (7) \exp\left(\frac{3}{2}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times) \quad (8) \exp\left(\frac{4}{2}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times)$$

or more succinctly if one has a preference $e^{\frac{1}{2}\pi \hat{k} \times}$, $e^{\frac{2}{2}\pi \hat{k} \times}$, $e^{\frac{3}{2}\pi \hat{k} \times}$, $e^{\frac{4}{2}\pi \hat{k} \times}$, $e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$, $e^{\pi(\frac{2}{2}\hat{k} \times + \hat{j} \times)}$, $e^{\pi(\frac{3}{2}\hat{k} \times + \hat{j} \times)}$, and $e^{\pi(\frac{4}{2}\hat{k} \times + \hat{j} \times)}$

$$\begin{array}{llll} (1) \exp\left(\frac{1}{2}\pi \hat{k} \times\right) & (2) \exp\left(\frac{2}{2}\pi \hat{k} \times\right) & (3) \exp\left(\frac{3}{2}\pi \hat{k} \times\right) & (4) \exp\left(\frac{4}{2}\pi \hat{k} \times\right) \\ (5) \exp\left(\pi\left(\frac{1}{2}\hat{k} \times + \hat{j} \times\right)\right) & (6) \exp\left(\pi\left(\frac{2}{2}\hat{k} \times + \hat{j} \times\right)\right) & (7) \exp\left(\pi\left(\frac{3}{2}\hat{k} \times + \hat{j} \times\right)\right) & (8) \exp\left(\pi\left(\frac{4}{2}\hat{k} \times + \hat{j} \times\right)\right) \end{array}$$

which one may appreciate are “full-glory” complex exponentials [and the theory of everything when coupled with a Gauss distributed Maxwell basis and Newton calculus]. This shows the group properties of combining what was known as complex exponential on three orthogonal Argand diagrams, with the very pleasing fact that it is all real in R^3 . No having to throw away the complex part. One important point is that the plus operator in the exponential is not commutative. Fairly obvious but needs to be borne in mind and it is not all that obvious when thinking in terms of conventional complex numbers where everything commutes.

$$r_4^2 = \left(\hat{k} \hat{k}' + \hat{k} \times\right)^2 \left(\hat{j} \hat{j}' + \hat{j} \times\right)^2 = \exp\left(\pi\left(\frac{2}{2}\hat{k} \times + \hat{j} \times\right)\right) = \exp\left(\pi \hat{i} \times\right) = \hat{i} \hat{i}' + \left(\hat{i} \times\right)^2 \text{ is minimal, and is the full rank } i\text{-hat-double-cross}$$

and obviously

$$r_4^4 = \left(\hat{k} \hat{k}' + \hat{k} \times\right)^4 \left(\hat{j} \hat{j}' + \hat{j} \times\right)^2 = \exp\left(\pi\left(\frac{4}{2}\hat{k} \times + \hat{j} \times\right)\right) = \exp\left(\pi \hat{j} \times\right) = \hat{j} \hat{j}' + \left(\hat{j} \times\right)^2$$

The set of 8 operators can be diadically expanded and these correspond one-for-one with matrix representations that spectroscopists and x-ray crystallographers have deduced from algebraic methods. This demonstrates definitively that these operators for dihedral groups are the basis-exponentials [full rank complex numbers]. As diadically expanded matrix operators, these are

for the **axis of proper rotation**

and for the **reflections with rotation**

$$(1) \sigma_4^1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(5) r_4^1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(2) \sigma_4^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(6) r_4^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(3) \sigma_4^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(7) r_4^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(4) \sigma_4^4 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_4 = I$$

$$(8) r_4^4 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^4 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

When operators are unique, and when operators are not unique, is also of relevance. This is relevant to the theory of dihedral groups, point groups, and Rubiks cubes, and linear algebra in general. The [principal] proper rotations are unique when minimal but for the reflections, there are alternatives depending on which principal axis is chosen to do the reflection. I'll discuss one example just to make the point for some interesting mathematics and show that the two are equivalent by mathematical rather than physical methods:

- Rotate around k -hat (with k -cross [k -curl]) then reflect around j -hat (with j -double-cross [j -div]) or
- Reflect around i -hat (with i -hat-double-cross [i -div]) then rotate around k -hat (with k -cross [k -curl])

In mathematical terms, the former is

$$\left(\hat{k} \hat{k}' + \hat{k} \times\right) \left(\hat{j} \hat{j}' + \hat{j} \times\right)^2 = \exp\left(\frac{1}{2}\pi \hat{k} \times\right) \exp\left(\pi \hat{j} \times\right) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = r_4^1$$

whilst the later is

$$\left(\hat{i} \hat{i}' + \hat{i} \times\right)^2 \left(\hat{k} \hat{k}' + \hat{k} \times\right) = \exp\left(\pi \hat{i} \times\right) \exp\left(\frac{1}{2}\pi \hat{k} \times\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = r_4^1$$

which are the same (r_4^1) hence are equivalent operators for D_4 [and also for Rubiks cube for bit tracking, although not for a Rubiks cube considered as a system]. This can also be in terms of non-principal axes on the plane. Note also the process of the above example in the opposite order for the reflections:

- Reflect around j -hat (with j -double-cross [j -div]) then rotate around k -hat (with k -cross [k -curl]) or
- Rotate around k -hat (with k -cross [k -curl]) then reflect around i -hat (with i -hat-double-cross [i -div])

$$(\hat{j} \hat{j}' + \hat{j} \times)^2 (\hat{k} \hat{k}' + \hat{k} \times) = \exp(\pi \hat{j} \times) \exp\left(\frac{1}{2} \pi \hat{k} \times\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = r_4^3$$

$$(\hat{k} \hat{k}' + \hat{k} \times)(\hat{i} \hat{i}' + \hat{i} \times)^2 = \exp\left(\frac{1}{2} \pi \hat{k} \times\right) \exp(\pi \hat{i} \times) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = r_4^3$$

which also shows that r_4^1 and r_4^3 are **skew-symmetric** [**negative transpose equivalent**, noting that the double-crosses are **invariant to transpose** and are **involutes** (since the square of the double-crosses is the identity)]. These arguments mean that the rotation elements, which are basis operators in the form of basis exponentials, can be manipulated mathematically [within a closed group system]. Hence, deriving the complete set of the D_4 character tables by mathematical means can make for an interesting task for students or researchers. The maths can also show [or prove] the various alternatives when a path to a given symmetry is not unique. The reflections can alternatively be in terms of the basis exponential of i -hat double-cross or any other axis on the plane appropriately rotated by k -cross. These then diadically expand into the same operators demonstrating the equivalence [as shown]. These matrices and matrix operations are best performed with a matrix capable calculator [since adding and multiplying **ones** and **negative ones** is a job for a calculator rather than a human].

D_4 character table

The character tables for D_4 are the second order tables. Since there are $2n = 8$ symmetry operators, the second order tables have $(2n)^2 = 64$ entries. One can get a square paper cut out and enumerate them, or just multiply the relevant two matrices [after all of the mathematical theory of producing those matrices is understood, that is, by reference to k -cross as the complex number and also, for higher understanding, as the unforced solution to a first order curled differential equation]. I'll take the second approach here, that of multiplying the two relevant matrices: The proper rotations and reflections with rotation are respectively $\sigma_4^1 \sigma_4^2 \sigma_4^3 \sigma_4^4$ and $r_4^1 r_4^2 r_4^3 r_4^4$

	$\sigma_4^4 = I$	σ_4^1	σ_4^2	σ_4^3	r_4^4	r_4^1	r_4^2	r_4^3
$\sigma_4^4 = I$	$\sigma_4^4 = I$	σ_4^1	σ_4^2	σ_4^3	r_4^4	r_4^1	r_4^2	r_4^3
σ_4^1	σ_4^1	σ_4^2	σ_4^3	$\sigma_4^4 = I$	r_4^1	r_4^2	r_4^3	r_4^4
σ_4^2	σ_4^2	σ_4^3	$\sigma_4^4 = I$	σ_4^1	r_4^2	r_4^3	r_4^4	r_4^1
σ_4^3	σ_4^3	$\sigma_4^4 = I$	σ_4^1	σ_4^2	r_4^3	r_4^4	r_4^1	r_4^2
r_4^4	r_4^4	r_4^3	r_4^2	r_4^1	$\sigma_4^4 = I$	σ_4^3	σ_4^2	σ_4^1
r_4^1	r_4^1	r_4^4	r_4^3	r_4^2	σ_4^1	$\sigma_4^4 = I$	σ_4^3	σ_4^2
r_4^2	r_4^2	r_4^1	r_4^4	r_4^3	σ_4^2	σ_4^1	$\sigma_4^4 = I$	σ_4^3
r_4^3	r_4^3	r_4^2	r_4^1	r_4^4	σ_4^3	σ_4^2	σ_4^1	$\sigma_4^4 = I$

I am not sure which notation is better: the informative form $e^{\pi(\frac{a}{2}\hat{k}\times + \frac{m}{2}\hat{j}\times + \frac{l}{2}\hat{i}\times)}$ which is consistent with, for example, Miller indices

$$e^{\frac{1}{2}\pi\hat{k}\times}, e^{\frac{2}{2}\pi\hat{k}\times}, e^{\frac{3}{2}\pi\hat{k}\times}, e^{\frac{4}{2}\pi\hat{k}\times}, e^{\pi(\frac{1}{2}\hat{k}\times + \frac{2}{2}\hat{j}\times)}, e^{\pi(\frac{2}{2}\hat{k}\times + \frac{2}{2}\hat{j}\times)}, e^{\pi(\frac{3}{2}\hat{k}\times + \frac{2}{2}\hat{j}\times)}, \text{ and } e^{\pi(\frac{4}{2}\hat{k}\times + \frac{2}{2}\hat{j}\times)}$$

or the simplest form

$$I, e^{\frac{1}{2}\pi\hat{k}\times}, e^{\pi\hat{k}\times}, e^{\frac{3}{2}\pi\hat{k}\times}, e^{\pi\hat{j}\times}, e^{\pi(\frac{1}{2}\hat{k}\times + \hat{j}\times)}, e^{\pi\hat{i}\times}, e^{\pi(\frac{3}{2}\hat{k}\times + \hat{j}\times)} \text{ (noting that } e^{\pi\hat{i}\times} = e^{\pi(\frac{3}{2}\hat{k}\times + \frac{2}{2}\hat{j}\times)} \text{ is the reflection [double-cross] around } i\text{-hat)}$$

I was going to use the simplest form but it is worthwhile to bear in mind that these are special forms of $e^{\pi(\frac{a}{2}\hat{k}\times + \frac{m}{2}\hat{j}\times + \frac{l}{2}\hat{i}\times)}$ so I have decided to instead use the Miller index form. Note that the form $e^{\pi(\frac{a}{2}\hat{k}\times + \frac{m}{2}\hat{j}\times + \frac{l}{2}\hat{i}\times)} = \mathbf{cisk}\left(\frac{\pi}{2}n\right)\mathbf{cisl}\left(\frac{\pi}{2}m\right)\mathbf{cisi}\left(\frac{\pi}{2}l\right)$ rotates by $\pi/2$ around k -hat n times, then around j -hat m times, and then around i -hat l times in that order since the plus in the exp- π doesn't commute. Other orderings (for 3 stacked layers of D_4 for a Rubiks cube for example) require long chains of exponentials, and as I have stated in several pasts contexts, the longest of which is expected [before simplifications from cyclic order properties cause repeats] to be that of

God's number for a Rubiks cube [again, which is 21]. A Rubiks cube is the fullest physical expression of full rank rectangular quaternions. This is because it is the simplest form that still contains a full group of all k -cross, j -cross, and i -cross, unlike D_4 which is only a subset and corresponds to the super-floppy which is an interesting but overly simple Rubiks derivative, that of $3 \times 3 \times 1$ (3by3by1).

I used to do an overwhelming lot of mathematics with the actuarial and statistical degrees and all, then again, with the PhD and the mathematics in the science degree, but now I hardly do any mathematics at all [had to again now with DSP (one may wonder if I am a sucker for punishment...)]. Because I am getting hassled so hard, I need to do everything with minimal energy so I have ended up doing almost everything on my programmable calculator and on my laptop, whilst writing [with lots of **cubing**, practicing the **Freidrich method** and **solving derivatives** (of the twisty puzzle variety)]. The diadically expanded matrices have worked like a dream.

	$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$
$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$
$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$
$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$
$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$
$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\frac{5}{2}\pi \hat{k} \times}$
$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\frac{4}{2}\pi \hat{k} \times} = I$
$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\frac{3}{2}\pi \hat{k} \times}$	$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\frac{1}{2}\pi \hat{k} \times}$
$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}$	$e^{\frac{5}{2}\pi \hat{k} \times}$	$e^{\frac{4}{2}\pi \hat{k} \times} = I$	$e^{\frac{1}{2}\pi \hat{k} \times}$	$e^{\frac{3}{2}\pi \hat{k} \times}$

$I, e^{\frac{1}{2}\pi \hat{k} \times}, e^{\pi \hat{k} \times}, e^{\frac{3}{2}\pi \hat{k} \times}, e^{\pi \hat{j} \times}, e^{\pi(\frac{1}{2}\hat{k} \times + \hat{j} \times)}, e^{\pi \hat{i} \times},$ and $e^{\pi(\frac{3}{2}\hat{k} \times + \hat{j} \times)}$

thinking of using $e\pi = e^\pi = 23.14$ as the constant

$$e\pi\left(\frac{3}{2}\hat{k} \times + \hat{j} \times\right)$$

The symmetry operators for the elements of the second order table, as diadically expanded matrices, are obtained by multiplying the relevant matrices (symmetry operators) together, one from the leading column one from the leading row. This is easily possible by hand but too easy to mess up the all important signs [computers handle signs well, humans don't as all four-unit maths (and linear algebra and DSP uni) students know all too well]. This is a nano-second calculation for a programmable calculator. I put this table together without hardly having to think at all, which I think is what good mathematics (with a good calculator) is all about.

	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Obviously, since it is a closed group, each operator occurs once in each column and once in each row within the table of second order products. Also observe that each symmetry matrix operator has three ones and six zeroes, similarly as for the discussion of k -cross. This is because they are rectangular [rotation matrices at $\pi/2$ and π for which the cosines are both zero or the sines are both zero]. These map, and hence permute, each component of a **position** or **colour** vector orthogonally and, once and only once. The curls (k -cross for example) have one negative for the curl whilst the divs (j -double-cross for example) have two negatives where the divs here correspond to reflections with a full set of ones along the diagonal [with one positive and two negatives]. Here, when there are curls on one axis with a div of another, there are also those with three negatives. Also, all of those that have been reflected (around i -hat or j -hat) have a negative value of the rotational eigenvalue of k -hat, which makes it possible to eyeball the symmetry matrix operator and tell what it is doing: if the kk' is negative then the operator reflects whilst if it is positive it is a proper rotation (rotation without reflection). This is also the sign of the one in the **trace** before the diagonal cosine is evaluated [known and well appreciated in linear algebra] to a one or zero.

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{k\text{-cross}} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{k\text{-triple-cross}} \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{\text{reflected } k\text{-cross}} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{\text{reflected } k\text{-triple-cross}}$$

[all (above), obviously, full rank]. The diagonal matrices (below) are the full rank double-crosses. **(1)** the positive entry on the diagonal corresponding to the basis variable that is crossed and **(2)** the two negative entries corresponding to the remaining basis hats, that is, to the orthogonal complement of the axis around which [hence the plane on which] the **position** or **colour** vector is rotated

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{identity}} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{k\text{-double-cross}} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{j\text{-double-cross}} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{i\text{-hat-double-cross}}$$

k -cross with the rotational entries (ij' and ji') signs changed (from $-ij'$ and ji' to ij' and $-ji'$) is obviously k -cross cubed (k -cross transpose), and the identity from either the fourth power of k -cross or the square of one of the double-crosses. This means now 6 of

the 8 of the matrices can be identified at sight and seen what the effect on a **position** or **colour** vector is. The remaining 2 are (1) the reflections with one quarter rotation and (2) reflection with three rotations, both with entries the same as k -cross but with a different arrangement of negatives. Both have the k -hat rotational eigenvalue negative (showing that it is a reflection from a j -hat or i -hat double-cross) and one of these has both values of the otherwise curl entries positive whilst the other both negative. One can look at it like an alphabet and language of nine entries of zeros and ones, 6 zeros and 3 ones, with the positions and signs of the ones determining the meaning of the operator, in this case for rotations and reflections of a square.

Each operator in the table of products occurs once in each row and once in each column. If each operator is assigned a value between 1 and 8 (because it is a $4 \times 2 = 8$ group), then the table will form a square where each row and each column sum to a common value. Hence, the table will form a **magic square**. In this case of D_4 , with magic number $1+2+\dots+8 = 36$. Similarly with all dihedral groups of regular polygons, each forming the even magic square corresponding to twice the numbers of vertices (twice the order of the dihedral group). The ordering of the operators in the binary product and hence the ordering of the rows and columns can be permuted to produce a great variety of squares for puzzles, for example, or grouped in different ways to produce special magic squares. This could also be used as a means to produce codes and ciphers.

Such a table for the closed group can be extended to the third order products which is then a cube. Each element is then reproduced once in each row, once in each column, and once in each depth. If each of these is assigned a number, then this will literally be the **cubic version of a magic square**, literally a **magic cube** [which Rubiks cube is sometimes called]. As it stands, it may seem that this probably wouldn't make for an enjoyable number puzzle since the depth cannot be expressed easily on the page, but the depth dimension could be tiled on the page and provide, for example, the magic properties of Sudoku where each tile also adds to the magic number. It would probably also provide additional sophistication [depth] to a code and/or cipher system. Interesting in-and-of-itself as well. This would be where depth is expressed as squares, so rows, columns, and squares form a cubic system of rows, columns, and depth. Since the first k -cross document, I have been trying to work out the full tensor system for the system tracking of the bits. One of the approaches I have been trying is laying out the cubic tensor system in a Sudoku grid which is showing significant promise, for example, keeping the depth dimension on the page, given that a Sudoku grid has a consistent number of bits for a k -hat, j -hat, and i -hat matrix basis expansion of a Rubiks cube if the inner mechanism is also considered a bit. Some reasoning is such: a Sudoku grid has $9 \times 9 = 81$ entries; there are two approaches I am considering taking for a Rubiks cube. The first of which, the Rubiks cube has $3 \times 3 \times 3 = 27$ bits and these can be tensor-matrix expanded in 3 ways, over k -hat, over j -hat, or over i -hat making $27 \times 3 = 81$ entries. Alternatively, expansion on say k -hat, the three layers of $3 \times 3 = 9$ entries can be Jordan blocks in a 9×9 Jordan form making 27 entries and 54 zeroes [81 altogether]... and then re-factored and expanded out on another basis hat axis as required... still working it out though.

The group symmetry operators are simulated physically with the middle k -hat layer of a Rubiks cube [or super-floppy] and mathematically with the relevant **colour vectors** for this layer, that is, the **colour vectors** for the 4 corners and 4 edges of the square. Very interestingly, one can do this for edges or corners as a bit tracking exercise. This correspond to the 4 planar faces (corresponding to the positive i -hat axis, the negative i -hat axis, the positive j -hat axis, and the negative j -hat axis) and the four middle edges of the outer i -hat and j -hat layers. The **colour vectors** therefore extend this theory considerably, since full orientation of the symmetry elements is introduced. This would be relevant if there where polarization elements in the corners or edges of the square planar geometry [for example if the geometry consisted of magnetic fields from Neodymium magnets].

For example, a polygon is considered a 2-dimensional object, but there is no such thing as a 2D object, the third dimension is hence still there but latently – either a **smaller order of magnitude** (a paper cutout), **in disguise** (perspective), or **just outright ignored** (complex algebra). A square piece of paper still has depth for example, as does a photograph with depth perception of the colour [but still 3D rather than 2D, colour playing tricks]. Hence, the depth of the polygon can be assigned a colour or a wavelength. The best assignment of colour is the standard colour scheme for a Rubiks cube. This then assigns a full orthogonal set of polarities, explicitly recognising depth which is ignored with a square paper cutout, for example, but implicitly by assigning a value or symbol to the 4 vertices on both sides. Why bother? Because this is what is required to translate time dynamics (in t) into spatial dynamics (in x , y , and z) as required for spectroscopy and molecular vibrations. Pretty interesting as a hobby too.

As a last word for this section discussing the dihedral group of order 4, that of a square, the single layer Rubiks derivative $3 \times 3 \times 1$ is boring [unsophisticated] compared to the full Rubiks $3 \times 3 \times 3$. How much more interesting the full cube is is how much more interesting and useful mathematics is going to get, as I said in the opening pages of the initial “ k -cross and the k -hat exponential”. Dihedral groups is an immediate example of this and the above working and logic for the D_4 group works for any dihedral group of arbitrary order by converting k -cross to the relevant roots of unity by Euler's formula (for the first subgroup of n elements for the pure rotations) and reflecting around any axis orthogonal to k -hat for the second set of n subgroup elements (rotations with reflection, so for the complete set of $2n$ symmetry elements).

Dihedral group of an equilateral triangle (trigonal planar) D_3

The D_3 group is an equilateral triangle and has cyclic order 6. The axis of pure rotation has subgroup cyclic order 3, whilst the mirror plane for reflection plus rotation has another subgroup addition of cyclic order 3, making the 6 in total. The pure rotation, similarly as all D_n , are k -crosses on the plane [formally considered an Argand plane] for the cubic roots of unity on that plane, the orthogonal complement of the k -hat axis. Ignoring the rotational eigenvalue, these are the k -hat basis exponential sampled at three points on the unit circle of the specified plane [the x - y plane for k -cross]. Hence, the $2n = 2 \times 3 = 6$ symmetry operators D_3 comprising the three rotations and three reflections with rotation for the planar equilateral triangle, that is, proper rotations $\sigma_3^1 \sigma_3^2 \sigma_3^3$ and reflections

$r_3^1 r_3^2 r_3^3$. The cube roots can't be put into simple basis-component form [zeros and ones for example] because they are not rectangular, that is, they are not rectangular quaternions [but rather a spanning orthogonal transformation of the rectangular system]. The fact that the basis components as operators can't be put in closed form [outside of here a cube root] adds more weight to the fact that rectangular basis components are the complex numbers **for the rectangular coordinate system** when appended with cross, the cross of vector cross product. Nevertheless, they can be expressed still as fractional powers of the full rank complex number expressed as basis components. For D_3 this is $(\hat{k} \hat{k}' + \hat{k} \times)^{4/3}$ which can also be expressed as the new formula with the eigenvalue outside the power $\hat{k} \hat{k}' + (\hat{k} \times)^{4/3}$. Situating a vertex and triangular bisector on the j -hat axis, the reflection rotation is then similarly as that for the square, a j -hat double-cross. Again, since j -hat is not latent, the rotational eigenvalue is required for the reflection. Therefore the full rank j -hat double-cross $\hat{j} \hat{j}' + \hat{j} \times \hat{j} \times$ is required rather than the singular counterpart $\hat{j} \times \hat{j} \times$. Note that these are j^2 considered as quaternions but here it is clear they must be considered as operators, again, $\times$ doing the trick of converting j -hat into a rotational operator and then adding $\hat{j} \hat{j}'$ and squaring for the full rank reflection. These set of operators for D_3 in terms of basis components are

$$\begin{array}{lll} (1) (\hat{k} \hat{k}' + \hat{k} \times)^{4/3} & (2) (\hat{k} \hat{k}' + \hat{k} \times)^{8/3} & (3) (\hat{k} \hat{k}' + \hat{k} \times)^{12/3} \\ (4) (\hat{k} \hat{k}' + \hat{k} \times)^{4/3} (\hat{j} \hat{j}' + \hat{j} \times)^2 & (5) (\hat{k} \hat{k}' + \hat{k} \times)^{8/3} (\hat{j} \hat{j}' + \hat{j} \times)^2 & (6) (\hat{k} \hat{k}' + \hat{k} \times)^{12/3} (\hat{j} \hat{j}' + \hat{j} \times)^2 \end{array}$$

where

$$(\hat{k} \hat{k}' + \hat{k} \times)^{12/3} = (\hat{k} \hat{k}' + \hat{k} \times)^4 = I = \hat{i} \hat{i}' + \hat{j} \hat{j}' + \hat{k} \hat{k}'$$

and

$$(\hat{k} \hat{k}' + \hat{k} \times)^{12/3} (\hat{j} \hat{j}' + \hat{j} \times)^2 = (\hat{k} \hat{k}' + \hat{k} \times)^4 (\hat{j} \hat{j}' + \hat{j} \times)^2 = I (\hat{j} \hat{j}' + \hat{j} \times)^2 = \hat{j} \hat{j}' + \hat{j} \times \hat{j} \times \quad (j\text{-double-cross with eigenvalue})$$

noting that geometric partitioning the full rank complex number $\hat{k} \hat{k}' + \hat{k} \times$ with fractional powers [here cube roots] rather than the basis exponential form, for better or worse, triviality or otherwise, is another way of expressing the magic of de-Moivre's theorem.

[wrote this paragraph before I obtained the arbitrary power formula: $(\hat{k} \hat{k}' + \hat{k} \times)^n = \hat{k} \hat{k}' + (\hat{k} \times)^n$] So it would be a possible academic

exercise to expand the binomial terms of $\hat{k} \times$ and the rotational eigenvalue $\hat{k} \hat{k}'$ with a fraction binomial index [4/3 for example] as a power series, collect and simplify and then refactor into the basis exponential which is whence closed up to the irrationality of e , demonstrating de-Moivre's theorem although this is probably a bit pedantic and is pretty much implied from all the various proofs regarding de-Moivre's theorem in the earlier documents whereby it is proved that exponentiation of singular k -cross gives the operator a rotational eigenvalue and hence makes it full-rank [in exactly the same way as adding the rotational eigenvalue directly].

Given that a binomial expansion for a non-integral index does not terminate, it was a bit disappointing that there wasn't a simple form of the non-rectangular basis components [there is now $(\hat{k} \hat{k}' + \hat{k} \times)^n = \hat{k} \hat{k}' + (\hat{k} \times)^n$] for arbitrary partitions of the unit circle other

than the square, that is, other than that which remains a right angle orthogonal system. On the other hand, it is very pleasing that these otherwise complicated expressions of raising the rectangular basis components to non-integral powers can easily be put in a simple closed expression of basis exponentials [thanks to de-Moivre's theorem... I had originally tried doing it with the singular k -cross similarly as for D_4]. This is essentially what Fourier transforms do, and now with k -cross, j -cross, and i -cross known to be the complex numbers, it will be possible to do this on all three of the orthogonal planes. Whilst I am still to work out the particulars, this is essential for spatial dynamics in R^3 . I am hoping that this will facilitate converting time series [which is ordinarily scalar as a parametric function of t] into spatial dynamics. This is my primary objective. I think why it hasn't yet been possible is because of the yet to be realised fact that there is a complex number (k -cross, j -cross, and i -cross) for each axis [in the context of the fundamental theorem of algebra and roots (poles and zeros) of polynomials of dynamics].

This also suggests that the Discrete Fourier Transform is essentially a transformed basis system, that is, a transformation of the two axes of the orthogonal complement of the rotation axis into multiple axes on that plane, and these in the theory of basis cross products as the complex numbers, suggests then that these are transformed sets of complex numbers. The significance of this will become clearer with another paper I am currently writing on polar-cylindrical and polar-spherical complex numbers, that is, basis cross products of these transformed coordinate systems. I have suggested in that one that it will be possible to explain all of this physically in terms of Rubiks derivatives. This is especially important in the context of x-ray crystallography since crystals are always a three-dimensional shape within an R^3 volume [rather than a scalar transfer function of time dynamics in t as for statistical time-series and DSP].

Looked at in this way, an n -point DFT can be seen to re-assign the 2 axes of the plane [4 if positive and negative are distinct] into $n/2$ axes [n if positive and negative are distinct] on that plane [the orthogonal complement to the spin axis], that is, assigns a set of new

transformed basis variables for the data for which it is calculated. This could be suggesting something higher order [that we are yet to recognise and exploit] but at minimum, it presents an **interpolation function** on that plane in terms of an **expanded out number of basis variables** for that plane [that has essentially **transformed and expanded out the basis**]. Time will tell if this leads to something that can be utilized to improve how the DFT is used and thought about [in addition to correlating the three orthogonal axes of a volume for spatial dynamics] but I have a feeling that it will. Examples of Rubiks derivatives with higher order planar geometry are the cheese cube (with two D_6 layers [but no centers]) and the YJ floppy ghost cube (with a single D_5 layer).

The basis-exponentials to which these correspond for D_3 , then, are the complex exponential sampled three times (over one 2π period), that is, the basis exponential for k -cross similarly as D_4 and that with the rotational eigenvalues added making the operators non-singular. Hence, the six **cisk** and **cisjj** symmetry operators are **(1-3) proper rotations** $\sigma_3^1 \sigma_3^2 \sigma_3^3$ with **cisk** and **(4-6) rotations with reflections** $r_3^1 r_3^2 r_3^3$ with **cisk + cisjj**. As basis exponentials these are

$$\begin{array}{lll} (1) \exp\left(\frac{2}{3}\pi \hat{k} \times\right) & (2) \exp\left(\frac{4}{3}\pi \hat{k} \times\right) & (3) \exp\left(\frac{6}{3}\pi \hat{k} \times\right) \\ (4) \exp\left(\frac{2}{3}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times) & (5) \exp\left(\frac{4}{3}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times) & (6) \exp\left(\frac{6}{3}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times) \end{array}$$

or more succinctly so long as one keeps in mind that the binary operator of $\times$ is non-commutative in the exponential

$$\begin{array}{lll} (1) \exp\left(\frac{2}{3}\pi \hat{k} \times\right) & (2) \exp\left(\frac{4}{3}\pi \hat{k} \times\right) & (3) \exp\left(\frac{6}{3}\pi \hat{k} \times\right) \\ (4) \exp\left(\pi\left(\frac{2}{3}\hat{k} \times + \hat{j} \times\right)\right) & (5) \exp\left(\pi\left(\frac{4}{3}\hat{k} \times + \hat{j} \times\right)\right) & (6) \exp\left(\pi\left(\frac{6}{3}\hat{k} \times + \hat{j} \times\right)\right) \end{array}$$

As diadically expanded matrix operators these are

for the **axis of proper rotation**

and for the **reflections with rotation**

$$\begin{array}{ll} (1) \sigma_3^1 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & (4) r_3^1 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ (2) \sigma_3^2 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & (5) r_3^2 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ (3) \sigma_3^3 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_4 = I & (6) r_3^3 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array}$$

where the proper rotations have a positive unity rotational eigenvalue for the k -hat axis whilst those that have been reflected this is negative. One of the vertices and bisectors of the equilateral triangle has been orientated on the j -hat axis for the rotation of reflection. This could have also been orientated on the i -hat axis. The diadically expanded operators differ in the arrangement of signs since the resulting positions of the vertices, in terms of **position** or **colour vectors**, differ in space because of the initial orientation and position in space after the π rotation to flip the triangle. Those above, for **(4)**, **(5)**, and **(6)** are **cisk(2 π /3)cisj(π)**, **cisk(4 π /3)cisj(π)**, and **cisk(6 π /3)cisj(π)**. The initial orientation of the triangle is what axis is chosen for the reflection and in what direction. Depending on how one chooses, for a k -hat axis of proper rotation these could alternatively be

$$\text{cisj}(\pi)\text{cisk}(2\pi/3), \quad \text{cisj}(\pi)\text{cisk}(4\pi/3), \quad \text{cisj}(\pi)\text{cisk}(6\pi/3)$$

or

$$\text{cisk}(2\pi/3)\text{cisi}(\pi), \quad \text{cisk}(4\pi/3)\text{cisi}(\pi), \quad \text{cisk}(6\pi/3)\text{cisi}(\pi)$$

or

$$\text{cisi}(\pi)\text{cisk}(2\pi/3), \quad \text{cisi}(\pi)\text{cisk}(4\pi/3), \quad \text{cisi}(\pi)\text{cisk}(6\pi/3)$$

D_3 Character Table

Being the second order products, character tables for D_3 have $2n \times 2n = 6 \times 6 = 36$ entries. These are most easily deduced by multiplying the relevant matrices. If one was to do the maths in terms of basis functions, then the multiplications could be done within cube roots.

	$\sigma_4^4=I$	σ_4^1	σ_4^2	r_4^4	r_4^1	r_4^2
$\sigma_4^4=I$	$\sigma_4^4=I$	σ_4^1	σ_4^2	r_4^4	r_4^1	r_4^2
σ_4^1	σ_4^1	σ_4^2	σ_4^3	r_4^1	r_4^2	r_4^3
σ_4^2	σ_4^2	σ_4^3	$\sigma_4^4=I$	r_4^2	r_4^3	r_4^4
r_4^4	r_4^4	r_4^3	r_4^2	$\sigma_4^4=I$	σ_4^3	σ_4^2
r_4^1	r_4^1	r_4^4	r_4^3	σ_4^1	$\sigma_4^4=I$	σ_4^3
r_4^2	r_4^2	r_4^4	r_4^4	σ_4^2	σ_4^1	$\sigma_4^4=I$

	$e^{\frac{6}{5}\pi i k n} = I$	$e^{\frac{1}{2}\pi i k n}$	$e^{\frac{2}{3}\pi i k n}$	$e^{\tau(\frac{4}{3} k n + j n)}$	$e^{\tau(\frac{1}{2} k n + j n)}$	$e^{\tau(\frac{2}{3} k n + j n)}$
$e^{\frac{6}{5}\pi i k n} = I$	$e^{\frac{6}{5}\pi i k n} = I$	$e^{\frac{1}{2}\pi i k n}$	$e^{\frac{2}{3}\pi i k n}$	$e^{\tau(\frac{4}{3} k n + j n)}$	$e^{\tau(\frac{1}{2} k n + j n)}$	$e^{\tau(\frac{2}{3} k n + j n)}$
$e^{\frac{1}{2}\pi i k n}$	$e^{\frac{1}{2}\pi i k n}$	$e^{\frac{1}{2}\pi i k n}$	$e^{\frac{2}{3}\pi i k n}$	$e^{\tau(\frac{1}{2} k n + j n)}$	$e^{\tau(\frac{2}{3} k n + j n)}$	$e^{\tau(\frac{4}{3} k n + j n)}$
$e^{\frac{2}{3}\pi i k n}$	$e^{\frac{2}{3}\pi i k n}$	$e^{\frac{2}{3}\pi i k n}$	$e^{\frac{6}{5}\pi i k n} = I$	$e^{\tau(\frac{2}{3} k n + j n)}$	$e^{\tau(\frac{4}{3} k n + j n)}$	$e^{\tau(\frac{1}{2} k n + j n)}$
$e^{\tau(\frac{4}{3} k n + j n)}$	$e^{\tau(\frac{4}{3} k n + j n)}$	$e^{\tau(\frac{2}{3} k n + j n)}$	$e^{\tau(\frac{1}{2} k n + j n)}$	$e^{\frac{6}{5}\pi i k n} = I$	$e^{\frac{1}{2}\pi i k n}$	$e^{\frac{2}{3}\pi i k n}$
$e^{\tau(\frac{1}{2} k n + j n)}$	$e^{\tau(\frac{1}{2} k n + j n)}$	$e^{\tau(\frac{4}{3} k n + j n)}$	$e^{\tau(\frac{2}{3} k n + j n)}$	$e^{\frac{1}{2}\pi i k n}$	$e^{\frac{6}{5}\pi i k n} = I$	$e^{\frac{1}{2}\pi i k n}$
$e^{\tau(\frac{2}{3} k n + j n)}$	$e^{\tau(\frac{2}{3} k n + j n)}$	$e^{\tau(\frac{1}{2} k n + j n)}$	$e^{\tau(\frac{4}{3} k n + j n)}$	$e^{\frac{2}{3}\pi i k n}$	$e^{\frac{1}{2}\pi i k n}$	$e^{\frac{6}{5}\pi i k n} = I$

[illegible]

I have the basis exponentials programmed in my TI, so I evaluated the k -hat exponential at $2\pi/3$ and the full rank j -hat double-cross (j -hat exponential at π) and then just multiplied the products out for that above making short work of the table, similarly as for D_4 .

One can also demonstrate for example that $(\hat{k} \hat{k}' + \hat{k} \times)^{4/3}$ is the basis form of the k -hat exponential at $2\pi/3$ if it is not immediately obvious from de-Moivre's theorem, as I did when I was working it out [obvious to me now but had to check that it all worked out after sorting out what was what since basis variables can play tricks (similarly as colour) with nilpotence]. I also have k -hat (the unit basis vector for the k -hat axis) programmed in the TI so, take this vector multiply it by its transpose and add the singular k -cross to obtain the full rank k -cross, and then raise to $4/3$. The rotation matrices are the basis-exponentials so the k -hat basis exponential is simply evaluated at $2\pi/3$. These are equivalent. This hence demonstrates the equivalence with the rotation matrix for, in this case, pure rotation of D_3 . [obvious but until they are checked...and above is how I did so].

Similar qualitative arguments can be made about the entries of the group table as for D_4 so that one can eyeball the matrix and see immediately what it is doing physically to the equilateral triangle of the D_3 group [that is, to the **position** or **colour** vector of a position and orientation on D_3].

Since D_3 is simple, there are only 2 structure forms when signs are ignored:

$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	The cube root of unity is the k-hat basis exponential at $2\pi/3$, the unit circle partitioned into three parts (rather than four parts for k-cross itself). This therefore has 120° and 240° cosines ($\pm \frac{1}{2}$) in the first two diagonal elements, 120° and 240° sines ($\pm \frac{\sqrt{3}}{2}$) in the curl components, and the unity rotational eigenvalue (k-hat cosines, cosk) in the last diagonal entry (the $\hat{k} \hat{k}'$ position since $(\text{cosk})\hat{k} = 1\hat{k}$).
--	--

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	The pure reflection element is the j-double-cross basis exponential [the full rank j-hat-double-cross], which is diagonal with negative elements corresponding to the i-hat and k-hat components (ii' and kk' positions) whilst a positive one corresponding to the j-hat component (jj' position) around which it rotates for the reflection. That is, reflection around j-hat [with the full rank j-hat-double-cross] changes the sign of i-hat and k-hat [of that operated on] whilst leaving the sign of j-hat unchanged. D_4 had all four of the identity, j-hat-double-cross, i-hat-double-cross, and k-hat-double-cross because the structure was orthogonal, D_3 only has the identity and j-hat-double-cross, the axis of pure reflection.
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The reflection of the first matrix in a second order product therefore changes the sign of the first and third columns (rows) of the second matrix whilst leaving the 2nd (middle) column (rows) unchanged [columns if right to left and rows if left to right]. The change of sign of the third column (row) corresponds to the change in the sign of the unitary k -hat rotational eigenvalue of the proper rotation [and hence why it is negative]. So similarly as for D_4 , a negative k -hat eigenvalue indicates the symmetry element involves the reflection.

All the matrices that are not diagonal [i.e., not either the identity or the full rank j -hat double-cross] are shuffling of the signs within the core k -hat basis exponential at $2\pi/3$, that is, halves on the first two diagonals (square root of a quarter) and halves of root threes (square root of three quarters) on the first two anti-diagonals. The positions and numbers of negative signs then determine its function.

Note that partitioning the unit circle for the order of the polygon ($2\pi/3$ for D_3 and $2\pi/4$ for D_4 for example, $2\pi/n$ for D_n in general) is analogous as partitioning the unit circle for a discrete Fourier transform and will be why these symmetry operators are used in x-ray crystallography together with the discrete Fourier transform (i.e., with Miller indices). This is also the central algorithm of DSP: almost always, one first takes the z -transform and then converts it to the Fourier transform by substituting the complex exponential for the linear z . This is exactly converting linearity into circles as discussed in my take on what I called generalized circles [where I have cleared up several of the theoretical difficulties encountered in electrical engineering (now that it can be said that k -cross is the complex number) noting that it will be similar for other faculties as well, although it is most relevant to electrical engineering since this vocation relies on the Fourier transform the most out of just about any vocation].

Dihedral group of a hexagon D_6

The dihedral group of a hexagon is important because it is the symmetry group of benzene although I will not compile the character tables here because the D_6 group has cyclic order 12, the character tables have $(2n)^2 = 144$ entries. The symmetry elements are the six corresponding to the axis of proper rotation plus six of these rotations with the reflection (reflection plus proper rotation in an appropriate order).

In basis notation, and despite not having closed forms other than in basis exponential form, these for interest and aesthetics are

(1) $I = (\hat{k} \hat{k}' + \hat{k} \times)^{0/6}$ (4) $(\hat{k} \hat{k}' + \hat{k} \times)^{12/6} = \hat{k} \hat{k}' + \hat{k} \times \hat{k} \times$ (7) $(\hat{j} \hat{j}' + \hat{j} \times)^2 = \hat{j} \hat{j}' + \hat{j} \times \hat{j} \times$	(2) $(\hat{k} \hat{k}' + \hat{k} \times)^{4/6}$ (5) $(\hat{k} \hat{k}' + \hat{k} \times)^{16/6}$ (8) $(\hat{k} \hat{k}' + \hat{k} \times)^{4/6} (\hat{j} \hat{j}' + \hat{j} \times)^2$	(3) $(\hat{k} \hat{k}' + \hat{k} \times)^{8/6}$ (6) $(\hat{k} \hat{k}' + \hat{k} \times)^{20/6}$ (9) $(\hat{k} \hat{k}' + \hat{k} \times)^{8/6} (\hat{j} \hat{j}' + \hat{j} \times)^2$
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$$(10) (\hat{k} \hat{k}' + \hat{k} \times)^2 (\hat{j} \hat{j}' + \hat{j} \times)^2$$

$$(11) (\hat{k} \hat{k}' + \hat{k} \times)^{16/6} (\hat{j} \hat{j}' + \hat{j} \times)^2$$

$$(12) (\hat{k} \hat{k}' + \hat{k} \times)^{20/6} (\hat{j} \hat{j}' + \hat{j} \times)^2$$

For the axis of proper rotation, the unit circle is partitioned into $n=6$ parts in the full rank k -cross, that is, in the k -hat basis exponential (equivalent to a six point DFT) giving the subgroup of the proper rotation. The partition of the circle is $2\pi/6$, i.e., $(\pi/3)n$ for $n = 0,1,2,3,4,5$. This gives the six subgroup elements of proper rotation. For the rotations with reflection, situating a vertex of the planar hexagon on the j -hat axis, and applying the same full rank j -hat double-cross (j -hat basis exponential at π) to reflect it (swapping the bottom face with the top face), and multiplying this to the six elements of proper rotation gives the subgroup of the remaining six elements of the group

$$(1) I = \exp\left(\frac{0}{3}\pi \hat{k} \times\right)$$

$$(2) \exp\left(\frac{1}{3}\pi \hat{k} \times\right)$$

$$(3) \exp\left(\frac{2}{3}\pi \hat{k} \times\right)$$

$$(4) \exp\left(\frac{3}{3}\pi \hat{k} \times\right)$$

$$(5) \exp\left(\frac{4}{3}\pi \hat{k} \times\right)$$

$$(6) \exp\left(\frac{5}{3}\pi \hat{k} \times\right)$$

$$(7) \exp(\pi \hat{j} \times)$$

$$(8) \exp\left(\frac{1}{3}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times)$$

$$(9) \exp\left(\frac{2}{3}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times)$$

$$(10) \exp\left(\frac{3}{3}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times)$$

$$(11) \exp\left(\frac{4}{3}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times)$$

$$(12) \exp\left(\frac{5}{3}\pi \hat{k} \times\right) \exp(\pi \hat{j} \times)$$

In diadically expanded operator form, noting the similarity to D_3 , those of the proper axis of rotation are

$$(1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(2) \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(3) \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(4) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(5) \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(6) \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

whilst those of the rotation plus reflection are

$$(7) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(8) \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(9) \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(10) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(11) \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(12) \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

noting that it is clear that D_3 is a subgroup of D_6 . What one may well appreciate is how this has made easy work of otherwise very complicated mathematics. Nothing abstract [myself being an applied mathematician], no complicated algebra, just dividing up circles on the relevant planes with basis exponentials. The character tables can be compiled either by programmable calculator or a few lines of MATLAB code.

Dihedral group of a pentagon D_5

I include the operators of the pentagon, the dihedral group of order 5, because even although the principles are the same as those previously the pentagon is a polygon face of one of the Platonic solids, the dodecahedron. Hence the equations will be required shortly so I enumerate them here. In basis notation,

$$(1) (\hat{k} \hat{k}' + \hat{k} \times)^{20/5}$$

$$(2) (\hat{k} \hat{k}' + \hat{k} \times)^{4/5}$$

$$(3) (\hat{k} \hat{k}' + \hat{k} \times)^{8/5}$$

$$(4) (\hat{k} \hat{k}' + \hat{k} \times)^{12/5}$$

$$(5) (\hat{k} \hat{k}' + \hat{k} \times)^{16/5}$$

$$(6) (\hat{k} \hat{k}' + \hat{k} \times)^{20/5} (\hat{j} \hat{j}' + \hat{j} \times)^2$$

$$(7) (\hat{k} \hat{k}' + \hat{k} \times)^{4/5} (\hat{j} \hat{j}' + \hat{j} \times)^2$$

$$(8) (\hat{k} \hat{k}' + \hat{k} \times)^{8/5} (\hat{j} \hat{j}' + \hat{j} \times)^2$$

$$(9) (\hat{k} \hat{k}' + \hat{k} \times)^{12/5} (\hat{j} \hat{j}' + \hat{j} \times)^2$$

$$(10) (\hat{k} \hat{k}' + \hat{k} \times)^{16/5} (\hat{j} \hat{j}' + \hat{j} \times)^2$$

which have the corresponding basis exponentials

$$\begin{aligned}
 (1) \exp\left(\frac{0}{5} \pi \hat{k} \times\right) & (2) \exp\left(\frac{2}{5} \pi \hat{k} \times\right) & (3) \exp\left(\frac{4}{5} \pi \hat{k} \times\right) & (4) \exp\left(\frac{6}{5} \pi \hat{k} \times\right) & (5) \exp\left(\frac{8}{5} \pi \hat{k} \times\right) \\
 (6) \exp\left(\pi \hat{j} \times\right) & (7) \exp\left(\frac{2}{5} \pi \hat{k} \times + \pi \hat{j} \times\right) & (8) \exp\left(\frac{4}{5} \pi \hat{k} \times + \pi \hat{j} \times\right) & (9) \exp\left(\frac{6}{5} \pi \hat{k} \times + \pi \hat{j} \times\right) & (10) \exp\left(\frac{8}{5} \pi \hat{k} \times + \pi \hat{j} \times\right)
 \end{aligned}$$

and after diadic expansion

$$\begin{pmatrix} \cos(n(2\pi/5)) & -\sin(n(2\pi/5)) & 0 \\ \sin(n(2\pi/5)) & \cos(n(2\pi/5)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad n = 0, 1, 2, 3, \dots, 4$$

$$\begin{pmatrix} \cos(n(2\pi/5)) & -\sin(n(2\pi/5)) & 0 \\ \sin(n(2\pi/5)) & \cos(n(2\pi/5)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -\cos(n(2\pi/5)) & -\sin(n(2\pi/5)) & 0 \\ -\sin(n(2\pi/5)) & \cos(n(2\pi/5)) & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad n = 0, 1, 2, 3, \dots, 4$$

Although sine and cosine at $2\pi/5$ are rational up to square roots, they are still algebraically large (compared to zeros and ones) making the matrices bulky so I have just kept these as general formulas in n . Note that sine and cosine at $2\pi/5$ are

$$\sin(2\pi/5) = \frac{\sqrt{2(\sqrt{5}+5)}}{4} \quad \text{and} \quad \cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$$

Trivial dihedral group of a line D_2

The trivial form of a polygon is a line segment. Hence, the trivial dihedral group is D_2 with $2n = 4$ group elements. What is interesting is that it is a combination of n orthogonal double-crosses (divs). If the line segment is on the $\hat{j}$ -axis and it rotates in two states around the $\hat{k}$ -axis, then the two double-crosses are $\hat{j}$ -hat-double-cross and $\hat{k}$ -hat-double-cross. The operators are identity, $\hat{k}$ -hat-double-cross, $\hat{j}$ -hat-double-cross, and $\hat{i}$ -hat-double-cross [$\hat{k}$ -double-cross times $\hat{j}$ -double-cross]. This is interesting for purposes of intuition.

The symmetry elements of D_2 , the identity and the double-crosses on each axis, are each contained in D_4 so D_2 is a subgroup of D_4 . Similarly D_2 and D_4 are subgroups of D_8 . Since these are groups with orders in factors of two, this may be of relevance for the Fast DFT algorithm. These are, by the same reasoning as D_4 , D_3 , D_6 , and D_5

$$(1) (\hat{k} \hat{k}' + \hat{k} \times)^0 \quad (2) (\hat{k} \hat{k}' + \hat{k} \times)^2 \quad (3) (\hat{k} \hat{k}' + \hat{k} \times)^0 (\hat{j} \hat{j}' + \hat{j} \times)^2 \quad (4) (\hat{k} \hat{k}' + \hat{k} \times)^2 (\hat{j} \hat{j}' + \hat{j} \times)^2$$

and simplifying for

$$(1) I \quad (2) (\hat{k} \hat{k}' + \hat{k} \times)^2 \quad (3) (\hat{j} \hat{j}' + \hat{j} \times)^2 \quad (4) (\hat{i} \hat{i}' + \hat{i} \times)^2$$

that is, the four elements of the D_2 group are the Identity and the set of basis-double-crosses

$$(1) I \quad (2) \hat{k} \hat{k}' + \hat{k} \times \hat{k} \times \quad (3) \hat{j} \hat{j}' + \hat{j} \times \hat{j} \times \quad (4) \hat{i} \hat{i}' + \hat{i} \times \hat{i} \times$$

which correspond to $\sigma_4^2 r_4^2 \sigma_4^4 r_4^4$ in D_4 . These correspond to the previous concept of quaternions $1, k^2 = -1, j^2 = -1$, and $i^2 = -1$ but are full rank. These are, from previously,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{identity}} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\hat{k}\text{-hat-double-cross}} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{\hat{j}\text{-hat-double-cross}} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{\hat{i}\text{-hat-double-cross}}$$

General dihedral group of N -sided polygon D_N

The general arguments are fairly obvious from the previous discussion up to this point. I'll state the formulae so that the general formulae are clear for reference. In basis notation, (1) the full rank k -cross (for rotations of both subgroups) and (2) j -double-cross (for the reflection subgroup)

$$(\hat{k} \hat{k}' + \hat{k} \times)^{n(4/N)} \quad n = 0, 1, 2, 3, \dots, N-1$$

$$(\hat{k} \hat{k}' + \hat{k} \times)^{n(4/N)} (\hat{j} \hat{j}' + \hat{j} \times)^2 \quad n = 0, 1, 2, 3, \dots, N-1$$

and since $\hat{k} \hat{k}' + \hat{k} \times = \exp\left(\frac{1}{2} \pi \hat{k} \times\right)$ the **N -point partitioned basis exponential** is $\left(\hat{k} \hat{k}' + \hat{k} \times\right)^{n(4/N)} = \left(\exp\left(\frac{1}{2} \pi \hat{k} \times\right)\right)^{n(4/N)}$

$$\exp\left(n \frac{2}{N} \pi \hat{k} \times\right) \quad n = 0, 1, 2, 3, \dots, N-1$$

$$\exp\left(n \frac{2}{N} \pi \hat{k} \times\right) \exp\left(\pi \hat{j} \times\right) \quad n = 0, 1, 2, 3, \dots, N-1$$

For rotation matrix form, by Euler's theorem of the **N -point partitioned basis exponentials** and subsequent diadic expansion

$$\sigma_N^n = \begin{pmatrix} \cos(n(2\pi/N)) & -\sin(n(2\pi/N)) & 0 \\ \sin(n(2\pi/N)) & \cos(n(2\pi/N)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad n = 0, 1, 2, 3, \dots, N-1$$

$$r_N^n = \begin{pmatrix} \cos(n(2\pi/N)) & -\sin(n(2\pi/N)) & 0 \\ \sin(n(2\pi/N)) & \cos(n(2\pi/N)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -\cos(n(2\pi/N)) & \sin(n(2\pi/N)) & 0 \\ \sin(n(2\pi/N)) & \cos(n(2\pi/N)) & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad n = 0, 1, 2, 3, \dots, N-1$$

noting how the j -double-cross that reflects around the j -hat axis changes the sign of the first and third columns (rows if left multiplied rather than right) whilst leaving the second unchanged. This is where the rotational eigenvalue for k -hat has had its sign changed showing immediately that it is a reflection, as discussed previously, and now here again for arbitrary N . This has the same effect if the j -hat-double-cross is applied from the left (rather than the right) with the difference that the first and third rows (rather than columns) have sign changes whilst the second row (rather than columns) remains unchanged.

As a last point on the discussion of k -cross for dihedral groups, because of the equivalence of the expressions for the full rank complex number, such as

$$\hat{k} \hat{k}' + \hat{k} \times = \exp\left(\frac{1}{2} \pi \hat{k} \times\right) = \begin{pmatrix} c_{\pi/2} & -s_{\pi/2} & 0 \\ s_{\pi/2} & c_{\pi/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

these then present extended ways to express de-Moivre's theorem by choosing and equating two of the equivalent expressions for the basis exponential raised to a power. There are currently ${}^6C_2 = 5+4+3+2+1 = 15$ different ways in total of choosing any two of these expressions [for basis exponentials] to represent raising **cisk** to powers. Some of which are tautologies whilst others are even more powerful mathematically than de-Moivre's theorem itself. There are tautologies because it is no longer clear which are exponentials (to equate complex exponential to the sum of cos and complex sin and raise to a power) since they all are [exponentials].

$$\begin{array}{ccc} \hat{k} \hat{k}' + (\hat{k} \times)^n & (\hat{k} \hat{k}' + \hat{k} \times)^n & \exp\left(\frac{n}{2} \pi \hat{k} \times\right) \\ \text{cosk}(n\pi/2) + (\hat{k} \times) \text{sink}(n\pi/2) & \begin{pmatrix} c_{n\pi/2} & -s_{n\pi/2} & 0 \\ s_{n\pi/2} & c_{n\pi/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} c_{n\pi/2} & 0 & 0 \\ 0 & c_{n\pi/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} + (\hat{k} \times) \begin{pmatrix} s_{n\pi/2} & 0 & 0 \\ 0 & s_{n\pi/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

In equation form, this is any of the lines given that each are equivalent [equivalent expressions for the basis exponential at $n\pi/2$]

$$\begin{aligned} \hat{k} \hat{k}' + (\hat{k} \times)^n &= (\hat{k} \hat{k}' + \hat{k} \times)^n \\ &= \exp\left(\frac{n}{2} \pi \hat{k} \times\right) \\ &= \text{cosk}(n\pi/2) + (\hat{k} \times) \text{sink}(n\pi/2) \\ &= \begin{pmatrix} c_{n\pi/2} & -s_{n\pi/2} & 0 \\ s_{n\pi/2} & c_{n\pi/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{n\pi/2} & 0 & 0 \\ 0 & c_{n\pi/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} + (\hat{k} \times) \begin{pmatrix} s_{n\pi/2} & 0 & 0 \\ 0 & s_{n\pi/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Platonic Solids

When one learns the theory of dihedral groups, usually a triangle or square is cut out or drawn on the page and the vertices labelled with letters, say a, b, c, and d. It is then rotated and flipped to all the various states. These letters are essentially the same as applying colours (radiation wavelengths). The colours on a cube set the orientation, similarly as assigning a, b, c, and d. These (the labels of a, b, c, and d) are essentially **colour vectors**. So all of the theory of the planar groups of polygons represent single layers which can be stacked, for starters, into a cube and then the theory further extended to each of the platonic solids. This is what I do here [because I am writing other documents, I only do this for a cube here]. Each of the qualitative arguments regarding axes of proper rotations and symmetry planes essentially amounts to enumerating all possible combinations of the full rank basis crosses, that is, all possible combinations of the basis exponentials of $\hat{k}$, $\hat{j}$, and $\hat{i}$ at the specified angles that segment the unit circle on each of the lattice planes. This then suggests an extended form of Fourier transform for use with Miller indices for x-ray crystallography [which I will discuss in later documents (if I get the theory going for it)].

Tetrahedral, Cubic, Octahedral, Dodecahedral and Icosahedral Symmetry: Cubic

I start with the cube first [I am only going to do the cube here, the other Platonic solids I will work out later] for the same reason as working out basis-crosses for the square first for dihedral groups of polygons, notably, that the axes of rotation are rectangular and orthogonal. The group order of the cube is 24. In linear algebra, this group is enumerated as symmetrical axes. An easier way, which gives equivalent results, is to simply take each of the six faces and apply the rotational k -cross subgroup, the quartic roots of that particular axis (positive and negative z , positive and negative y , positive and negative x). Six faces times four positions of rotation of each is $6 \times 4 = 24$, the order of the cube.

From “**additions to $i = k \times (\cdot)$ k -cross & the k -hat exponential**”

“In basis format, enumeration of each of the 26 unique colour labeled bits as (unprimed column) **colour vectors** is”

“6 faces: right $\hat{i} \mathbf{r}$, left $-\hat{i} \mathbf{o}$, front $\hat{j} \mathbf{b}$, back $-\hat{j} \mathbf{g}$, top $\hat{k} \mathbf{y}$, bottom $-\hat{k} \mathbf{w}$ ”

Hence to move each face to the top, the following cross-products are required:

- | | |
|--|---|
| (1) Yellow top: $\mathbf{y}$ is at home at $\hat{k}$ -hat so keep unchanged with the identity, | I |
| (2) White top: $\mathbf{w}$ is at home at neg- $\hat{k}$ -hat so rotate around either i -hat or j -hat with (say) j -double-cross, | $(\hat{j} \hat{j}' + \hat{j} \times)^2$ |
| (3) Red top: $\mathbf{r}$ is at home at i -hat so rotate around j -hat with j -cross, | $\hat{j} \hat{j}' + \hat{j} \times$ |
| (4) Orange top: $\mathbf{o}$ is at home at neg- i -hat so rotate around j -hat with j -triple-cross (j -cross-conjugate), | $(\hat{j} \hat{j}' + \hat{j} \times)^3$ |
| (5) Blue top: $\mathbf{b}$ is at home at j -hat so rotate around i -hat with i -cross, | $\hat{i} \hat{i}' + \hat{i} \times$ |
| (6) Green top: $\mathbf{g}$ is at home at neg- j -hat so rotate around i -hat with i -triple-cross (i -cross-conjugate), | $(\hat{i} \hat{i}' + \hat{i} \times)^3$ |

Applying the full rank quartic roots of unity $\hat{k} \hat{k}' + \hat{k} \times$ to each of these six top orientations gives the 24 rotation operators for the cube

- | | | | |
|---|---|---|---|
| (1) $I = (\hat{k} \hat{k}' + \hat{k} \times)^4$ | $(\hat{k} \hat{k}' + \hat{k} \times)$ | $(\hat{k} \hat{k}' + \hat{k} \times)^2$ | $(\hat{k} \hat{k}' + \hat{k} \times)^3$ |
| (2) $(\hat{j} \hat{j}' + \hat{j} \times)^2$ | $(\hat{j} \hat{j}' + \hat{j} \times)^2 (\hat{k} \hat{k}' + \hat{k} \times)$ | $(\hat{j} \hat{j}' + \hat{j} \times)^2 (\hat{k} \hat{k}' + \hat{k} \times)^2$ | $(\hat{j} \hat{j}' + \hat{j} \times)^2 (\hat{k} \hat{k}' + \hat{k} \times)^3$ |
| (3) $(\hat{j} \hat{j}' + \hat{j} \times)$ | $(\hat{j} \hat{j}' + \hat{j} \times) (\hat{k} \hat{k}' + \hat{k} \times)$ | $(\hat{j} \hat{j}' + \hat{j} \times) (\hat{k} \hat{k}' + \hat{k} \times)^2$ | $(\hat{j} \hat{j}' + \hat{j} \times) (\hat{k} \hat{k}' + \hat{k} \times)^3$ |
| (4) $(\hat{j} \hat{j}' + \hat{j} \times)^3$ | $(\hat{j} \hat{j}' + \hat{j} \times)^3 (\hat{k} \hat{k}' + \hat{k} \times)$ | $(\hat{j} \hat{j}' + \hat{j} \times)^3 (\hat{k} \hat{k}' + \hat{k} \times)^2$ | $(\hat{j} \hat{j}' + \hat{j} \times)^3 (\hat{k} \hat{k}' + \hat{k} \times)^3$ |
| (5) $(\hat{i} \hat{i}' + \hat{i} \times)$ | $(\hat{i} \hat{i}' + \hat{i} \times) (\hat{k} \hat{k}' + \hat{k} \times)$ | $(\hat{i} \hat{i}' + \hat{i} \times) (\hat{k} \hat{k}' + \hat{k} \times)^2$ | $(\hat{i} \hat{i}' + \hat{i} \times) (\hat{k} \hat{k}' + \hat{k} \times)^3$ |
| (6) $(\hat{i} \hat{i}' + \hat{i} \times)^3$ | $(\hat{i} \hat{i}' + \hat{i} \times)^3 (\hat{k} \hat{k}' + \hat{k} \times)$ | $(\hat{i} \hat{i}' + \hat{i} \times)^3 (\hat{k} \hat{k}' + \hat{k} \times)^2$ | $(\hat{i} \hat{i}' + \hat{i} \times)^3 (\hat{k} \hat{k}' + \hat{k} \times)^3$ |

In basis exponential form,

- | | | | |
|---|---|---|---|
| (1) $\exp\left(\frac{4}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{1}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{2}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{3}{2} \pi \hat{k} \times\right)$ |
| (2) $\exp\left(\frac{1}{2} \pi \hat{j} \times\right)$ | $\exp\left(\frac{1}{2} \pi \hat{j} \times\right) \exp\left(\frac{1}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{1}{2} \pi \hat{j} \times\right) \exp\left(\frac{2}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{1}{2} \pi \hat{j} \times\right) \exp\left(\frac{3}{2} \pi \hat{k} \times\right)$ |
| (3) $\exp\left(\frac{3}{2} \pi \hat{j} \times\right)$ | $\exp\left(\frac{3}{2} \pi \hat{j} \times\right) \exp\left(\frac{1}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{3}{2} \pi \hat{j} \times\right) \exp\left(\frac{2}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{3}{2} \pi \hat{j} \times\right) \exp\left(\frac{3}{2} \pi \hat{k} \times\right)$ |
| (4) $\exp\left(\frac{1}{2} \pi \hat{i} \times\right)$ | $\exp\left(\frac{1}{2} \pi \hat{i} \times\right) \exp\left(\frac{1}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{1}{2} \pi \hat{i} \times\right) \exp\left(\frac{2}{2} \pi \hat{k} \times\right)$ | $\exp\left(\frac{1}{2} \pi \hat{i} \times\right) \exp\left(\frac{3}{2} \pi \hat{k} \times\right)$ |

$$\begin{array}{llll}
(5) \exp\left(\frac{3}{2}\pi \hat{i} \times\right) & \exp\left(\frac{3}{2}\pi \hat{i} \times\right) \exp\left(\frac{1}{2}\pi \hat{k} \times\right) & \exp\left(\frac{3}{2}\pi \hat{i} \times\right) \exp\left(\frac{2}{2}\pi \hat{k} \times\right) & \exp\left(\frac{3}{2}\pi \hat{i} \times\right) \exp\left(\frac{3}{2}\pi \hat{k} \times\right) \\
(6) \exp\left(\pi \hat{j} \times\right) & \exp\left(\pi \hat{j} \times\right) \exp\left(\frac{1}{2}\pi \hat{k} \times\right) & \exp\left(\pi \hat{j} \times\right) \exp\left(\frac{2}{2}\pi \hat{k} \times\right) & \exp\left(\pi \hat{j} \times\right) \exp\left(\frac{3}{2}\pi \hat{k} \times\right)
\end{array}$$

For putting the diadically expanded basis component into rotation matrix form, I first listed

$$I = \quad kk^4 \quad jj \quad jj^3 \quad ii \quad ii^3 \quad jj^2$$

and then

$$I = \quad kk^4 \quad kk^1 \quad kk^2 \quad kk^3$$

then multiplied each of these in a similar way as outer product

$$\begin{array}{l}
I \{kk^4, kk^1, kk^2, kk^3\} \\
jj \{kk^4, kk^1, kk^2, kk^3\} \\
jj^3 \{kk^4, kk^1, kk^2, kk^3\} \\
ii \{kk^4, kk^1, kk^2, kk^3\} \\
ii^3 \{kk^4, kk^1, kk^2, kk^3\} \\
jj^2 \{kk^4, kk^1, kk^2, kk^3\}
\end{array}$$

Noting that

$$\begin{aligned}
I, jj, jj^3, ii, ii^3, jj^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
I, kk^1, kk^2, kk^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Then

$$\begin{array}{llll}
(1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(2) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(3) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(4) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(5) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(6) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}$$

which multiply out to the required diadically expanded basis exponentials (combinations of rotation matrices) for the cube

$$(1) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
(2) \quad & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
(3) \quad & \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
(4) \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \\
(5) \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\
(6) \quad & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned}$$

which fully enumerates and classifies the 24 rectangular quaternion operators for R^3 . These can be classified in terms of structure whereby structure means in this context the positions of the zeros and ones, which are then further differentiated by the number and positions of positives and negatives

$$\begin{aligned}
(1) \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{identity and reflection structure} \\
(2) \quad & \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} & j\text{-cross structure} \\
(3) \quad & \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & k\text{-cross structure} \\
(4) \quad & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} & j\text{-cross-}k\text{-cross structure} \\
(5) \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} & i\text{-cross structure} \\
(6) \quad & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} & k\text{-cross-}j\text{-cross structure}
\end{aligned}$$

There are 6 structural forms: diagonal, anti-diagonal, and the other four forms which I do not have a good naming convention for yet, two of them are the k -cross structure and j -cross structure [and possibly k -cross- j -cross structure and k -cross- j -cross structure but this implies 2nd order whilst these have the same but rotated structure as k -cross and j -cross which are 1st order so those names aren't ideal... possibly jk -cross structure and kj -cross structure?]. Each structural form has 6 negatives in total.

I was hoping to quantize the basis and put it into a cubic tensor form although this has proved to be easier said than done [still think I will eventually be able to do so though].